Which magnetic fields support a zero mode?

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Joint work with

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Remembrance of things past

Hydrogenic Atom in a magnetic field

$$H_d = (-i\nabla - A(x))^2 - \frac{Z}{|x|}$$

acts on wavefunctions $\psi(x) \in \mathbb{C}$

$$B(x) = \mathrm{curl} A(x)$$

Denote the ground state energy by $E_d(Z, B)$

Kato's Diamagnetic inequality

$$|(-i\nabla - A(x))\psi(x)|^2 \geq |\nabla|\psi(x)||^2$$

$$E_d(Z, B) \geq E_d(Z, 0) \text{ , gauge invariance!}$$

Diamagnetism

This is not the whole truth: **Electrons carry spin**
Hydrogenic atom in a magnetic field

$$H = (-i\nabla - A(x))^2 - \sigma \cdot B(x) - \frac{Z}{|x|}$$

Pauli operator acts on **spinors**

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \ , \ \psi_1, \psi_2 \in \mathbb{C} \ , \ \int_{\mathbb{R}^3} |\psi_1|^2 + |\psi_2|^2 dx = 1$$

$$B = \mathrm{curl} A \ , \ \sigma \cdot B = \begin{pmatrix} B_3 & B_1 - iB_2 \\ B_1 + iB_2 & -B_3 \end{pmatrix}$$

For B large

$$E(Z, B) = \inf_{\|\psi\|=1} (\psi, H\psi) \to -\infty$$

The Hamiltonian is gauge invariant and ground state energy depends on B
Paramagnetism

The ground state energy should be compensated by the field energy

$$\mathcal{E}(\psi, A) = \|\sigma\cdot(-i\nabla - A)\psi\|^2 - Z\int_{\mathbb{R}^3} \frac{|\psi|^2}{|x|}dx + \frac{1}{8\pi\alpha^2}\int_{\mathbb{R}^3} |B(x)|^2 dx\ .$$

$$\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}\ , \text{Sommerfeld Fine Structure Constant}$$

$$[\sigma \cdot (-i\nabla - A(x))]^2 = (-i\nabla - A(x))^2 - \sigma \cdot B(x)$$

$$E(Z) = \inf_B \{E(Z, B) + \frac{1}{8\pi\alpha^2}\int_{\mathbb{R}^3} |B(x)|^2 dx\} = \inf_{\psi, A} \mathcal{E}(\psi, A) > -\infty\ ?$$

Zero Modes

$$\sigma \cdot (-i\nabla - A(x))\psi(x) = 0 \tag{1}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma \cdot (-i\nabla - A(x))$$

$$= \begin{pmatrix} -i\partial_3 - A_3 & -i\partial_1 - \partial_2 - (A_1 - iA_2) \\ -i\partial_1 + \partial_2 - (A_1 + iA_2) & +i\partial_3 + A_3 \end{pmatrix}$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \ , \ \psi_1, \psi_2 \in \mathbb{C} \ , \ \int_{\mathbb{R}^3} |\psi_1|^2 + |\psi_2|^2 dx = 1$$

$$B = \mathrm{curl} A \in L^2(\mathbb{R}^3 : \mathbb{R}^3)$$

$$\sigma \cdot (-i\nabla - A(x))\psi(x) = 0$$

$$\int_{\mathbb{R}^3} |\psi|^2 dx = 1 \ , \int_{\mathbb{R}^3} |B(x)|^2 dx < \infty$$

$$A_\lambda(x) = \lambda A(\lambda x) \ , \psi_\lambda(x) = \lambda^{3/2}\psi(\lambda x)$$

$$\sigma \cdot (-i\nabla - A_\lambda(x))\psi_\lambda(x) = 0$$

$$\mathcal{E}(\psi_\lambda, A_\lambda) = \lambda(-Z \int_{\mathbb{R}^3} \frac{|\psi|^2}{|x|} dx + \frac{1}{8\pi\alpha^2} \int_{\mathbb{R}^3} |B(x)|^2 dx)$$

Hydrogen is unstable if and only if there is a zero mode
(Fröhlich, Lieb, Loss 1986)

Zero modes (Loss-Yau, 1986)

$$\psi = \frac{1 + i\sigma \cdot x}{(1 + |x|^2)^{3/2}}\phi \ , \phi \text{ a constant spinor} \tag{2}$$

$$A(x) = \frac{3}{(1 + |x|^2)^2} \left((1 - |x|^2)\omega + 2(\omega \cdot x)x + 2\omega \wedge x\right)$$

$$\omega = \langle \phi, \sigma\phi \rangle \ .$$

$$\langle \phi, \psi \rangle = \overline{\phi_1}\psi_1 + \overline{\phi_2}\psi_2 \ , \text{inner product on } \mathbb{C}^2$$

Field lines are pull back of the Hopf fibration on $\mathbb{S}^3$ under stereographic projection

Sobolev's inequality

$$f : \mathbb{R}^d \to \mathbb{C}$$

$$\int_{\mathbb{R}^d} |\nabla f(x)|^2 dx \geq S \left(\int_{\mathbb{R}^d} |f(x)|^p dx \right)^{2/p}$$

$$p = \frac{2d}{d-2} \ ,$$

$$S_d = \frac{d(d-2)}{4} \ |\mathbb{S}^d|^{2/d} \ ,$$

and there is equality if and only if up to translations

$$f(x) = \left(c_\star \lambda + \frac{|x|^2}{\lambda} \right)^{-(d-2)/2} \quad , \lambda > 0$$

Rodemich (1966), Talenti (1976), Aubin (1975)

Measure of a potential supporting a bound state with negative energy

$$-\Delta - V(x)\ ,\ V(x) \geq 0\ ,\ V(x) \to 0\ |x| \to \infty\ .$$

If one seeks this measure in terms of an L^p norm of the potential, the only possibility is

$$\|V\|_{3/2}\ ,\ \text{scaling}\ V(x) \to \lambda^2 V(\lambda x)$$

$$\int_{\mathbb{R}^3} |\nabla f|^2 dx - \int_{\mathbb{R}^3} V|f|^2 dx \geq \|f\|_6^2 (S_3 - \|V\|_{3/2})$$

Necessary condition for a bound state is that $\|V\|_{3/2} > S_3$.
This condition is sharp!

A **simple** necessary condition for zero modes:

$$\sigma(-i\nabla - A)\psi = 0 \Rightarrow [\sigma(-i\nabla - A)]^2\psi = (-i\nabla - A)^2\psi - \sigma \cdot B\psi = 0$$

$$\|(-i\nabla - A)\psi\|_2^2 = \int_{\mathbb{R}^3} B(x) \cdot \langle\psi, \sigma\psi\rangle(x)dx \leq \|B\|_{3/2}\|\psi\|_6^2$$

$$\langle\psi, \sigma\psi\rangle| = |\psi| = \sqrt{\langle\psi, \psi\rangle}$$

by the diamagnetic inequality

$$\|(-i\nabla - A)\psi\|_2^2 \geq \|\nabla|\psi|\|_2^2$$

hence a necessary condition for a zero mode is

$$\|B\|_{3/2} \geq S_3$$

We may assume that $A \in L^3(\mathbb{R}^3 : \mathbb{R}^3)$.

$$A(x) = \mathrm{const.} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \wedge B(x) dx$$

satisfies

$$\mathrm{curlA} = B \ , \mathrm{div} A = 0$$

The estimate follows from the Hardy-Littlewood-Sobolev inequality

$$\| \frac{1}{|x|^\lambda} \star f \|_p \leq C_{p,d} \| f \|_q \ , \frac{1}{q} + \frac{\lambda}{d} = 1 + \frac{1}{p}$$

Theorem

Fix $3/2 < p < \infty$ *and let* $\psi \in L^p(\mathbb{R}^3 : \mathbb{C}^2)$ *be a solution of the zero mode equation* (1). *Then* $\psi \in L^r(\mathbb{R}^3 : \mathbb{C}^2)$ *for any* $3/2 < r < \infty$.

A not so simple improvement

$$\sigma \cdot (-i\nabla - A(x))\psi(x) = 0 \ , A \in L^3(\mathbb{R}^3;\mathbb{R}^3) \ , \qquad (3)$$
$$B = \mathrm{curl} A \in L^{3/2}(\mathbb{R}^3;\mathbb{R}^3) \ .$$

Theorem

Let $B \in L^{3/2}(\mathbb{R}^3 : \mathbb{R}^3)$ be a magnetic field, i.e., $\mathrm{div}\, B = 0$. A necessary condition for (3) to have a weak solution $0 \not\equiv \psi \in L^p(\mathbb{R}^3 : \mathbb{C}^2)$ for some $3/2 < p < \infty$ is that

$$\|B\|_{3/2} \geq 2S_3 \ .$$

Absence of eigenvalues for the Pauli-Operator is a difficult problem Cossetti, Fanelli, and Krejčiřík (2020)

Diamagnetic inequality

$$\psi = e^{iS}|\psi|$$

$$|(-i\nabla - A(x))\psi(x)|^2 = |\nabla|\psi||^2 + |\nabla S(x) - A(x)|^2|\psi|^2$$
$$\geq |\nabla|\psi||^2 \ .$$

Has interesting consequences

$$\left|(-i\nabla - A(x))^{-2}(x,y)\right| \leq \frac{1}{4\pi|x-y|}$$

Lemma

Let $\psi \in L^p(\mathbb{R}^3 : \mathbb{C}^2), p > 3/2$ *satisfy* $\sigma \cdot (-i\nabla - A)\psi = 0$. *Then* $\psi \in H^1(\mathbb{R}^3 : \mathbb{C}^2)$, *and* $|\psi| \in H^1(\mathbb{R}^3)$ *as well and moreover, almost everywhere in* $\mathbb{R}^3$,

$$|\nabla|\psi||^2 \leq \frac{2}{3}\,|(-i\nabla - A)\psi|^2 \ .$$

David M. J. Calderbank, Paul Gauduchon, and Marc Herzlich (2000)
Paul M. N. Feehan (2001)
Kato-Yau inequality
Elias Stein

An algebraic fact

$$v \in \mathbb{C}^2 \text{ , spinor , } \beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3 \text{ .}$$

$$(\Pi(\beta \otimes v))_j = \beta_j v - \frac{1}{3}\sigma_j \sigma \cdot \beta v \qquad \text{for } j = 1, 2, 3 \,.$$

$$\sum_{j=1}^{3} \sigma_j \, (\Pi(\beta \otimes v))_j = 0 \text{ , } \sigma_j^2 = I \text{ , } j = 1, 2, 3$$

$$(\beta \otimes v, \gamma \otimes w) = \sum_j \beta_j \gamma_j \langle v, w \rangle$$

$$(\Pi(\beta \otimes v), \gamma \otimes w)) = (\beta \otimes v, \Pi(\gamma \otimes w))$$

$$|\Pi(\beta \otimes v)|^2 = \frac{2}{3}|\beta|^2 |v|^2 \,.$$

$$|\psi|_\varepsilon = \sqrt{|\psi|^2 + \varepsilon^2}$$

$$\partial_j |\psi|_\varepsilon = \mathrm{Re}\left\langle \frac{\psi}{|\psi|_\varepsilon}, \partial_j \psi \right\rangle = \mathrm{Re}\left\langle \frac{\psi}{|\psi|_\varepsilon}, (\partial_j - iA_j)\,\psi \right\rangle$$

$$|\nabla|\psi|_\varepsilon| = \frac{\nabla|\psi|_\varepsilon}{|\nabla|\psi|_\varepsilon|} \cdot \nabla|\psi|_\varepsilon = \mathrm{Re}\left\langle \frac{\psi}{|\psi|_\varepsilon} \frac{\nabla|\psi|_\varepsilon}{|\nabla|\psi|_\varepsilon|}, (\nabla - iA)\,\psi \right\rangle \ . \quad (4)$$

$$(\nabla - iA)_j\,\psi = (\nabla - iA)_j\,\psi - \frac{1}{3}\sigma_j \sigma \cdot (\nabla - iA)\,\psi = [\Pi\,(\nabla - iA)\,\psi]_j\,,$$

$$|\nabla|\psi|_\varepsilon| = \mathrm{Re}\left\langle \frac{\psi}{|\psi|_\varepsilon} \frac{\nabla|\psi|_\varepsilon}{|\nabla|\psi|_\varepsilon|}, \Pi\,(\nabla - iA)\,\psi \right\rangle \ .$$

$$|\nabla|\psi|_\varepsilon| = \mathrm{Re}\left\langle \Pi\left(\frac{\psi}{|\psi|_\varepsilon}\frac{\nabla|\psi|_\varepsilon}{|\nabla|\psi|_\varepsilon|}\right), (\nabla - iA)\,\psi\right\rangle .$$

$$|\nabla|\psi|_\varepsilon| \leq \left|\Pi\left(\frac{\psi}{|\psi|_\varepsilon}\frac{\nabla|\psi|_\varepsilon}{|\nabla|\psi|_\varepsilon|}\right)\right| |(\nabla - iA)\,\psi| \ .$$

$$|\Pi(\beta \otimes v)|^2 = \frac{2}{3}|\beta|^2|v|^2 .$$

$$v = \psi/|\psi|_\varepsilon$$

$$\beta = \nabla|\psi|_\varepsilon/|\nabla|\psi|_\varepsilon|$$

ε tend to zero yields the claimed inequality.

The improvement on the diamagnetic inequality is not the whole story. We suspect that the Loss-Yau zero modes are the optimizers.

$$\psi = \frac{1 + i\sigma \cdot x}{(1+|x|^2)^{3/2}}\phi \;, \phi \text{ a constant spinor}$$

$$|\psi|^{1/2} = (1+|x|^2)^{-1/2}$$

is an optimizer for the Sobolev inequality

By the way: A straightforward computation shows that

$$\|B_{LY}\|_{3/2} = 4S_3$$

which is still off by a factor of 2 !!

Lemma

For any $\psi \in H^1(\mathbb{R}^3 : \mathbb{C}^2)$ *and any* $\varepsilon > 0$, *the function* $|\psi|_\varepsilon^{1/2}$ *is weakly differentiable with* $\nabla|\psi|_\varepsilon^{1/2} \in L^2(\mathbb{R}^3)$ *and one has almost everywhere and in the sense of* L^1,

$$\left|\nabla|\psi|_\varepsilon^{1/2}\right|^2 = \frac{1}{2}\mathrm{Re}\left\langle \nabla\frac{\psi}{|\psi|_\varepsilon}, \nabla\psi \right\rangle - \frac{1}{4}\left(2|\psi|_\varepsilon^{-1}|\nabla\psi|^2 - 3\frac{|\psi|^2}{|\psi|_\varepsilon^3}\left|\nabla|\psi|\right|^2\right). \quad (5)$$

The proof is a straightforward computation with Sobolev functions

$$\left|\nabla|\psi|_\varepsilon^{1/2}\right|^2 =$$

$$\frac{1}{2}\mathrm{Re}\left\langle(\nabla - iA)\frac{\psi}{|\psi|_\varepsilon}, (\nabla - iA)\psi\right\rangle$$

$$-\frac{1}{4|\psi|_\varepsilon}\left(2\left|(\nabla - iA)\psi\right|^2 - 3\frac{|\psi|^2}{|\psi|_\varepsilon^2}\left|\nabla|\psi|\right|^2\right)$$

$$= \frac{1}{2}\mathrm{Re}\left\langle(\nabla - iA)\frac{\psi}{|\psi|_\varepsilon}, (\nabla - iA)\psi\right\rangle$$

$$-\frac{1}{4|\psi|_\varepsilon}\left(2\left|(\nabla - iA)\psi\right|^2 - 3\left|\nabla|\psi|\right|^2\right) - \frac{3\varepsilon^2}{4|\psi|_\varepsilon^3}\left|\nabla|\psi|\right|^2 .$$

By Lemma 3

$$2\left|(\nabla - iA)\psi\right|^2 - 3\left|\nabla|\psi|\right|^2 \geq 0$$

and hence

$$\left|\nabla|\psi|_\varepsilon^{1/2}\right|^2 \leq \frac{1}{2}\mathrm{Re}\left\langle(\nabla - iA)\frac{\psi}{|\psi|_\varepsilon}, (\nabla - iA)\psi\right\rangle$$

$$\int_{\mathbb{R}^3}\left|\nabla|\psi|_\varepsilon^{1/2}\right|^2 dx \leq \frac{1}{2}\mathrm{Re}\int_{\mathbb{R}^3}\left\langle(\nabla - iA)\frac{\psi}{|\psi|_\varepsilon}, (\nabla - iA)\psi\right\rangle dx$$

$$= \frac{1}{2}\int_{\mathbb{R}^3} B(x)\cdot\frac{\langle\psi, \sigma\psi\rangle}{|\psi|_\varepsilon}dx$$

$$\leq \frac{1}{2}\int_{\mathbb{R}^3}|B(x)|\frac{|\psi|^2}{|\psi|_\varepsilon}dx$$

Letting $\varepsilon \to 0$

$$\int_{\mathbb{R}^3} \left|\nabla|\psi|^{1/2}\right|^2 dx \leq \frac{1}{2}\int_{\mathbb{R}^3} |B(x)||\psi| dx \quad (\star)$$

$$S_3\|\psi\|_3 \leq \frac{1}{2}\|B\|_{3/2}\|\psi\|_3$$

yields the result

Is this result optimal?

$$\psi = \frac{1}{\sqrt{2}r^{3/2}} \begin{pmatrix} \sqrt{r+z} \\ \frac{x+iy}{\sqrt{r+z}} \end{pmatrix} ,$$

$$\sigma \cdot \vec{x}\psi = r\psi$$

and

$$|\psi|^2 = \frac{1}{r^2} .$$

Now consider the monopole A-field

$$A = g\frac{(-y, x, 0)}{r(r+z)}$$

$$\mathrm{curl} A = g\frac{\vec{x}}{r^3} .$$

We also have that

$$\sigma{\cdot}A\psi = ig\frac{1}{\sqrt{2}r^{5/2}} \begin{pmatrix} -\frac{r-z}{\sqrt{r+z}} \\ \frac{x+iy}{\sqrt{r+z}} \end{pmatrix} , (-i)\sigma{\cdot}\nabla\psi = i\frac{1}{2}\frac{1}{\sqrt{2}r^{5/2}} \begin{pmatrix} -\frac{r-z}{\sqrt{r+z}} \\ \frac{x+iy}{\sqrt{r+z}} \end{pmatrix}$$

$$\sigma \cdot [(-i\nabla) - A]\psi = 0$$

if we choose $g = \frac{1}{2}$

$$|\nabla|\psi|^{1/2}|^2 = \frac{1}{4}\frac{1}{|x|^3}$$

$$\frac{1}{2}\frac{\langle\psi, B\cdot\sigma\psi\rangle}{|\psi|} = \frac{1}{4}\frac{1}{|x|^3}$$

This ‘saturates’ $(\star)$.

Euler-Lagrange equation for the Sobolev constant

$$-\Delta f - f^5 = 0$$

Regular solution up to translation

$$f_r(x) = 3^{1/4}\left(\frac{1}{\lambda} + \lambda x^2\right)^{-1/2}$$

Singular solution up to translation

$$f_s(x) = \frac{1}{4}|x|^{-1/2}$$

The function $|x|^{-1/2}$ is 'optimizer' for Hardy's inequality

$$\int_{\mathbb{R}^3} |\nabla f(x)|^2 dx \geq \frac{1}{4}\int_{\mathbb{R}^3} \frac{|f(x)|^2}{|x|^2} dx$$

$$\int_{\mathbb{R}^3} |\nabla f(x)|^2 dx - \int_{\mathbb{R}^3} V(x) f(x)^2 dx$$

$$\geq \int_{\mathbb{R}^3} |\nabla f^*(x)|^2 dx - \int_{\mathbb{R}^3} V^*(x) f^*(x)^2 dx$$

$$\geq \int_{\mathbb{R}^3} |\nabla f^*(x)|^2 dx - \int_{\mathbb{R}^3} \frac{|f^*(x)|^2}{|x|^2} dx \sup_x \{|x|^2 V^*(x)\}$$

A necessary condition for the existence of a negative bound state is

$$|\mathbb{S}^3|^{-2/3} \|V\|_{w,3/2} = \sup_x \{|x|^2 V^*(x)\} > \frac{1}{4}$$

Theorem

Let $B \in L^{3/2}_w(\mathbb{R}^3 : \mathbb{R}^3)$ *be a magnetic field, i.e.,* $\operatorname{div} B = 0$*. If* (1) *has a weak solution* $0 \not\equiv \psi \in H^1(\mathbb{R}^3 : \mathbb{C}^2)$*, then*

$$\|B\|_{w,3/2} \geq \frac{1}{2}\left(\frac{4\pi}{3}\right)^{2/3}$$

or, equivalently,

$$\sup_{x\in\mathbb{R}^3} |x|^2|B|^*(x) \geq \frac{1}{2}\ . \tag{6}$$

The monopole field $\frac{1}{2}\frac{x}{|x|^3}$ is an optimzer

Theorem

Assume that $\lambda \in L^3(\mathbb{R}^3)$ *is a real function. If the equation*

$$-i\sigma \cdot \nabla\psi = 3\lambda(x)\psi$$

has a weak solution $0 \not\equiv \psi \in L^p(\mathbb{R}^3 : \mathbb{C}^2)$ *for some* $3/2 < p < \infty$*, then*

$$\frac{1}{4}|\mathbb{S}^3|^{\frac{2}{3}} \leq [\int |\lambda(x)|^3 dx]^{2/3} \ . \tag{7}$$

There is equality in (7) *if*

$$\lambda(x) = \frac{1}{1+x^2} \ ,$$

in which case (2) *is a solution.*

$$-i\sigma\cdot\nabla\psi = 3\lambda(x)\psi \ ,$$

we consider again the operator

$$\Pi(\beta\otimes\psi)_j = \beta_j\psi - \frac{1}{3}\sigma_j\sigma\cdot\beta\psi$$

but proceed in a slightly different manner by considering

$$\Pi(\partial_j - i\lambda(x)\sigma_j)\phi = (\partial_j - i\lambda(x)\sigma_j)\phi - \frac{1}{3}\sigma_j\sigma\cdot(\nabla - i\lambda(x)\sigma)\phi \ .$$

Lemma

Let $\psi\in L^p(\mathbb{R}^3:\mathbb{C}^2), p>3/2$ satisfy $-i\sigma\cdot\nabla\psi = 3\lambda(x)\psi$. Then $\psi\in H^1(\mathbb{R}^3:\mathbb{C}^2)$, and $|\psi|\in H^1(\mathbb{R}^3)$ as well and moreover, almost everywhere in $\mathbb{R}^3$,

$$3\,|\nabla|\psi||^2 \le 2\,|[\nabla - i\lambda(x)\sigma]\psi|^2 \ . \tag{8}$$

With similar computations as before we find

$$\int_{\mathbb{R}^3} \left|\nabla|\psi|^{1/2}\right|^2 dx \leq 3 \int_{\mathbb{R}^3} \lambda^2 |\psi| dx \ .$$

$$S_3 \leq 3 \left(\int_{\mathbb{R}^3} |\lambda|^3 dx \right)^{2/3}$$

$$\lambda(x) = \frac{1}{1+|x|^2}$$

saturates this inequality and with

$$\psi = \frac{1 + i\sigma \cdot x}{(1+|x|^2)^{3/2}} \phi$$

we find

$$-i\sigma \cdot \nabla \psi = \frac{3}{1+|x|^2} \psi \ .$$

The results for the zero modes in magnetic fields hold in any odd dimension.

In the case where the potential is a scalar function the results hold in all dimension ≥ 2.

If we assume that $\lambda(x) > 0$ and the solution ϕ smooth, the sharp estimate follows from Hijazi's inequality: Dirac operator D on $\mathbb{S}^d$ (metric g)

$$g_u = e^{2u} g \ , D_u \psi_u = \left(e^{-\frac{d+1}{2}u} D e^{\frac{d-1}{2}u} \psi \right)_u$$

Hijazi (1991)

$$\lambda_1(D_u)^2 \geq \frac{d}{4(d-1)} \lambda_1(L_u) \ ,$$

$$L_u = e^{-\frac{(d+2)}{2}u} L e^{\frac{(d-2)}{2}u} \ .$$

L conformal Laplacian on $\mathbb{S}^d$

THANK YOU