New results on the Lieb-Thirring inequality

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The nonlinear Schrödinger equation for orthonormal functions II. Application to Lieb-Thirring inequalities. Comm. Math. Phys. (2021), arXiv:2002.04964.
The periodic Lieb-Thirring inequality, ArXiv e-prints (2020). To Ari Laptev on the occasion of his 70th birthday, arXiv:2010.02981.

Como-Milano-Napoli Mathematical Physics Online Seminar, Feb. 8th, 2021

Gagliardo-Nirenberg-Sobolev

$$H^1(\mathbb{R}^d) = \left\{f \in L^2(\mathbb{R}^d)\ :\ \nabla f \in L^2(\mathbb{R}^d)\right\} \hookrightarrow L^p(\mathbb{R}^d) \text{ for all } \begin{cases} 2 \leq p \leq \infty & \text{if } d = 1 \\ 2 \leq p < \infty & \text{if } d = 2 \\ 2 \leq p \leq \frac{2d}{d-2} & \text{if } d \geq 3 \end{cases}$$

$$\left(\int_{\mathbb{R}^d} |u(x)|^p\, dx\right)^{\frac{4}{d(p-2)}} \leq C^{\rm GNS}_{p,d} \left(\int_{\mathbb{R}^d} |u(x)|^2\, dx\right)^{\frac{(2-d)p+2d}{d(p-2)}} \int_{\mathbb{R}^d} |\nabla u(x)|^2\, dx$$

Optimizer **unique** up to translations, dilations and multiplication by a phase, given by (radial positive) solution to **nonlinear Schrödinger equation (NLS)**

$$-\Delta Q - Q^{p-1} + Q = 0$$

At $p = \frac{2d}{d-2}$ (Sobolev), $Q(x) = (1+|x|^2)^{\frac{2-d}{2}}$ solves $-\Delta Q = Q^{p-1}$ (Emden-Fowler)

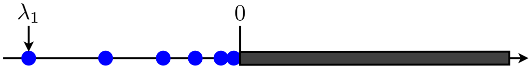
NLS important in applications: Bose-Einstein condensation, nonlinear optics, water waves, Langmuir waves in plasmas, etc

Gagliardo '59, Nirenberg '59, Sobolev '63, Strauss '77, Gidas-Ni-Nirenberg '81, Berestycki-Lions '83, Weinstein '83, Coffman '72, Kwong '89, McLeod '93

Spectral interpretation

Introduce the **dual variable** V to the function $|u|^2 \in L^{\frac{p}{2}}(\mathbb{R}^d,\mathbb{R})$

$$V \in L^{\gamma+\frac{d}{2}}(\mathbb{R}^d,\mathbb{R}), \quad \gamma + \frac{d}{2} = \left(\frac{p}{2}\right)' \Longleftrightarrow \gamma = 1 - \frac{d}{2} + \frac{2}{p-2} \begin{cases} \geq \frac{1}{2} & \text{for } d=1 \\ > 0 & \text{for } d=2 \\ \geq 0 & \text{for } d \geq 3 \end{cases}$$



$\sigma\big(-\Delta + V(x)\big)$

Lemma (GNS $\Leftrightarrow$ spectral bound)

The lowest (negative) eigenvalue of the self-adjoint operator $-\Delta + V(x)$ satisfies

$$\big|\lambda_1(-\Delta+V)\big|^\gamma \leq L^{(1)}_{\gamma,d} \int_{\mathbb{R}^d} V(x)_-^{\gamma+\frac{d}{2}}\, dx$$

with the best constant $L^{(1)}_{\gamma,d} = \left(\frac{2\gamma}{2\gamma+d}\right)^{\gamma+\frac{d}{2}} \left(\frac{d}{2\gamma}\right)^{\frac{d}{2}} \left(C^{GN}_{p,d}\right)^{\frac{d}{2}}$ (including the case $\gamma = 0$).

Proof. Variational principle $\lambda_1(-\Delta+V) = \inf\left\{\int_{\mathbb{R}^d} |\nabla u|^2 + V|u|^2 \ : \ \int_{\mathbb{R}^d} |u|^2 = 1\right\}$

$$\int_{\mathbb{R}^d} |\nabla u|^2 + V|u|^2 \geq \|\nabla u\|^2_{L^2} - \|V_-\|_{L^{\gamma+\frac{d}{2}}} \|u\|^2_{L^p} \geq \|\nabla u\|^2_{L^2} - \|V_-\|_{L^{\gamma+\frac{d}{2}}} \left(C^{\rm GN}_{p,d}\|\nabla u\|^2_{L^2}\right)^{\frac{d(p-2)}{2p}}$$

Lieb-Thirring inequality

$$\sum_{n=1}^{N} |\lambda_n(-\Delta+V)|^\gamma \leq L^{(N)}_{\gamma,d} \int_{\mathbb{R}^d} V(x)_-^{\gamma+\frac{d}{2}}\,dx, \qquad \gamma \begin{cases} \geq \frac12 & \text{for } d=1 \\ >0 & \text{for } d=2 \\ \geq 0 & \text{for } d\geq 3 \end{cases}$$

with the best constant $L^{(N)}_{\gamma,d} \leq N L^{(1)}_{\gamma,d}$, non-decreasing with N

Theorem (Lieb-Thirring '75–76)

We have $\lim_{N\to\infty} L^{(N)}_{\gamma,d} = L_{\gamma,d} < \infty$ *for all* γ *as before, hence*

$$\sum_{n=1}^{\infty} |\lambda_n(-\Delta+V)|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V(x)_-^{\gamma+\frac{d}{2}}\,dx$$

- γ not an end point by Lieb-Thirring '75–76
- $\gamma = 0$ in $d \geq 3$ by Cwikel-Lieb-Rozenblum '72–77
- $\gamma = 1/2$ in $d = 1$ by Weidl '96
- inequality is **extensive**, think of $V = v + v(\cdot - R)$

Main application: large fermionic systems, used by Lieb-Thirring '75 to give a simplified proof of the stability of matter (Dyson-Lenard '67)

R.L. Frank, The Lieb-Thirring inequalities: Recent results and open problems, arXiv:2007.09326 (2020)

Lowest kinetic energy at fixed marginal

For $\Psi \in L^2((\mathbb{R}^d)^N, \mathbb{C})$ with $|\Psi|^2$ symmetric and $\|\Psi\|_{L^2} = 1$, define the marginal

$$\mu_\Psi(x) = \int_{(\mathbb{R}^d)^{N-1}} |\Psi(x, x_2, ..., x_N)|^2 \, dx_2 \cdots dx_N.$$

Theorem (Hoffman-Ostenhof '77)

$$\int_{\mathbb{R}^{dN}} |\nabla \Psi|^2 \geq N \int_{\mathbb{R}^d} |\nabla \sqrt{\mu_\Psi}(x)|^2 \, dx$$

with equality if and only if $\Psi(x_1, ..., x_N) = e^{i\theta} \prod_{n=1}^N \sqrt{\mu_\Psi(x_n)}$.

Theorem (Lieb-Thirring '76)

If Ψ is ***anti-symmetric****, we have*

$$\int_{\mathbb{R}^{dN}} |\nabla \Psi|^2 \geq c_d^{LT} N^{1+\frac{2}{d}} \int_{\mathbb{R}^d} \mu_\Psi(x)^{1+\frac{2}{d}} \, dx.$$

Proof. This is the **dual of LT** at $\gamma = 1$:

$$\int_{\mathbb{R}^{dN}} |\nabla \Psi|^2 - N \int_{\mathbb{R}^d} \mu_\Psi V = \left\langle \Psi, \left(\sum_{n=1}^N -\Delta_{x_n} - V(x_n) \right) \Psi \right\rangle$$

$$\geq \lambda_1 \left(\sum_{n=1}^N -\Delta_{x_n} - V(x_n) \right) = \sum_{n=1}^N \lambda_n(-\Delta - V) \geq -L_{1,d} \int_{\mathbb{R}^d} V_+^{1+\frac{d}{2}}$$

Scenarii for the best constant $L_{\gamma,d}$

$$\sum_{n=1}^{N} |\lambda_n(-\Delta+V)|^\gamma \leq L^{(N)}_{\gamma,d} \int_{\mathbb{R}^d} V(x)_-^{\gamma+\frac{d}{2}}\,dx, \qquad \gamma \begin{cases} \geq \frac12 & \text{for } d=1 \\ >0 & \text{for } d=2 \\ \geq 0 & \text{for } d\geq 3 \end{cases}$$

1. $L_{\gamma,d} = L^{(N)}_{\gamma,d}$ attained for $V \in L^{\gamma+\frac{d}{2}}(\mathbb{R}^d)$ with $N<\infty$ **eigenvalues**
2. $L_{\gamma,d}$ attained for $V \in L^{\gamma+\frac{d}{2}}(\mathbb{R}^d)$ with **infinitely many eigenvalues**
3. $L_{\gamma,d}$ "attained" for $V \in L^{\gamma+\frac{d}{2}}_{\rm loc}(\mathbb{R}^d) \setminus L^{\gamma+\frac{d}{2}}(\mathbb{R}^d)$ with **both sides being infinite**

$$\lim_{R\to\infty} \frac{\displaystyle\sum_{n=1}^{\infty} |\lambda_n(-\Delta+V\mathbb{1}_{B_R})|^\gamma}{\displaystyle\int_{B_R} V(x)_-^{\gamma+\frac{d}{2}}\,dx} \overset{?}{=} L_{\gamma,d}$$

4. $L_{\gamma,d}$ not attained in any reasonable sense

Known results:

1. happens at $\gamma \in \{\frac12,\frac32\}$ in $d=1$ where $L_{\gamma,1} = L^{(1)}_{\gamma,1}$

   Lieb-Thirring '76, Hundertmark-Lieb-Thomas '98

3. happens for all $\gamma \geq \frac32$ in $d\geq 1$ with $V \equiv -\text{cnst}$ (semi-classical)

   Lieb-Thirring '76, Aizenman-Lieb '78, Laptev-Weidl '00

Lieb-Thirring conjecture

For $V \equiv -\mu < 0$, semi-classics gives

$$\frac{\sum_{n=1}^{\infty} \left|\lambda_n(-\Delta - \mu \mathbb{1}_{B_R})\right|^\gamma}{\int_{B_R} V(x)_-^{\gamma+\frac{d}{2}}\,dx} \underset{R\to\infty}{\sim} \frac{\iint_{\mathbb{R}^{2d}} (|p|^2 - \mu \mathbb{1}_{B_R}(x) \leq 0)_-^\gamma \frac{dx\,dp}{(2\pi)^d}}{\mu^{\gamma+\frac{d}{2}}|B_R|} = \int_{\mathbb{R}^d} (|p|^2-1)_-^\gamma \frac{dp}{(2\pi)^d} =: L^{\rm sc}_{\gamma,d}$$

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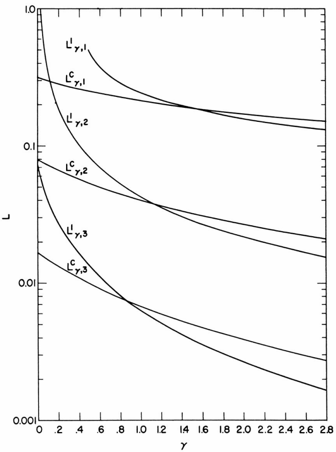


Conjecture (Lieb-Thirring '76)

We have $L_{\gamma,d} = \max\left(L^{(1)}_{\gamma,d}, L^{\rm sc}_{\gamma,d}\right)$.

Situation now **known to be more complicated**

For instance, $L_{\gamma,d} > L^{\rm sc}_{\gamma,d}$ for $\gamma < 1$

Helffer-Robert '90

Optimal potential cannot have $N < \infty$ for $\gamma > \max\{0, 2 - \frac{d}{2}\}$

$$\sum_{n=1}^{N} |\lambda_n(-\Delta + V)|^\gamma \leq L^{(N)}_{\gamma,d} \int_{\mathbb{R}^d} V(x)_-^{\gamma+\frac{d}{2}}\, dx, \qquad \gamma \begin{cases} \geq \frac{1}{2} & \text{for } d = 1 \\ > 0 & \text{for } d = 2 \\ \geq 0 & \text{for } d \geq 3 \end{cases}$$

Theorem (Non-optimality of finite N, Frank-Gontier-ML '21)

- *Let $d \geq 1$ and $\gamma > \max\left\{0, 2 - \frac{d}{2}\right\}$. There exists a infinite sequence $N_1 = 1 < N_2 = 2 < N_3 < \cdots$ such that $L^{(N_j-1)}_{\gamma,d} < L^{(N_j)}_{\gamma,d}$. In particular, $L_{\gamma,d} > L^{(N)}_{\gamma,d}$ for all $N \in \mathbb{N}$.*
- *For any $N = N_j$, there exists an optimal potential $V_N < 0$ for $L^{(N)}_{\gamma,d}$. Choosing the normalization $\int_{\mathbb{R}^d} |V_N|^{\gamma+\frac{d}{2}} = 1$, the corresponding eigenfns solve the NLS-type system*

$$\Bigg(-\Delta - \underbrace{\left(\frac{2\gamma}{L^{(N)}_{\gamma,d}(d+2\gamma)} \sum_{n=1}^{N} |\lambda_n|^{\gamma-1} |u_n|^2\right)^{\frac{1}{\gamma+\frac{d}{2}-1}}}_{-V_N}\Bigg) u_k = \lambda_k\, u_k, \qquad k = 1, \ldots, N \qquad (1)$$

d	0		$\frac{1}{2}$		1		$\frac{3}{2}$	
1			|		?		|	$N = \infty$
2			?		|		$N = \infty$	
3		?	|		$N = \infty$			
≥ 4			$N = \infty$					

Sketch of proof

Lemma 1 (existence and decay)

Let $\gamma > 1/2$ in $d = 1$ and $\gamma > 0$ in $d \geq 2$. If $L^{(N)}_{\gamma,d} > L^{(N-1)}_{\gamma,d}$ then $L^{(N)}_{\gamma,d}$ admits an optimizer $V_N \in L^{\gamma+\frac{d}{2}}(\mathbb{R}^d)$, with $\lambda_N < \lambda_{N+1} \leq 0$. It is a real analytic function which decays exponentially fast at infinity. The corresponding eigenfunctions solve the NLS system (1).

Only $\gamma \geq 1$ in paper (use of dual inequality), general case in preparation. Uses concentration-compactness (Lions '84), weak continuity of eigenvalues under tight convergence of potentials

Lemma 2 (binding)

Assume that $\gamma > \max\left\{0, 2 - \frac{d}{2}\right\}$. If $L^{(N)}_{\gamma,d}$ admits an optimizer then $L^{(2N)}_{\gamma,d} > L^{(N)}_{\gamma,d}$.

Proof of the theorem:

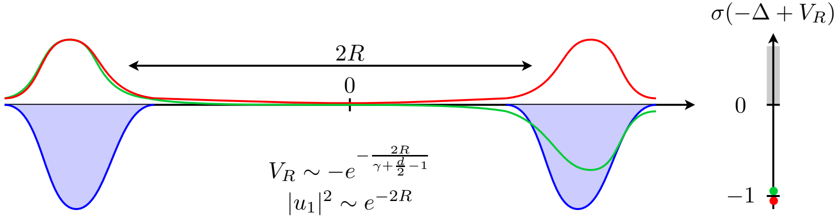
- $L^{(1)}_{\gamma,d}$ has the NLS optimizer Q, hence $L^{(2)}_{\gamma,d} > L^{(1)}_{\gamma,d}$ by Lemma 2
- $L^{(2)}_{\gamma,d}$ has an optimizer by Lemma 1
- $L^{(4)}_{\gamma,d} > L^{(2)}_{\gamma,d}$ by Lemma 2
- either $L^{(3)}_{\gamma,d} > L^{(2)}_{\gamma,d}$ then take $N_3 = 3$, or $L^{(4)}_{\gamma,d} > L^{(3)}_{\gamma,d} = L^{(2)}_{\gamma,d}$ then take $N_3 = 4$
- iterate

Proof of $L^{(2N)}_{\gamma,d} > L^{(N)}_{\gamma,d}$ (Lemma 2) for $N = 1$

$$-\Delta Q - Q^{p-1} + Q = 0$$

For $V_R(x) = -\Big(Q(x - Re_1)^2 + Q(x + Re_1)^2\Big)^{\frac{1}{\gamma+\frac{d}{2}-1}}$

- recall that $Q(x) \underset{|x|\to\infty}{\sim} C|x|^{\frac{1-d}{2}} e^{-|x|}$
- estimate the exponentially small deviation of the eigenvalues
- due to **nonlinear quantum tunnelling** between the two wells
- competition with orthonormalization of $Q(\cdot - Re_1), Q(\cdot + Re_1)$ which generates a 2nd order error $\sim e^{-4R}$
- favorable for $2 + \frac{2}{\gamma+\frac{d}{2}-1} < 4 \iff \gamma > 2 - \frac{d}{2}$



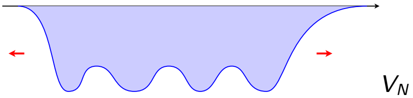
Statistical mechanics point of view

$$\left(-\Delta - c\left(\sum_{n=1}^{N} |\lambda_n|^{\gamma-1}|u_n|^2\right)^{\frac{1}{\gamma+\frac{d}{2}-1}}\right)u_k = \lambda_k\, u_k, \qquad k = 1,\ldots,N$$

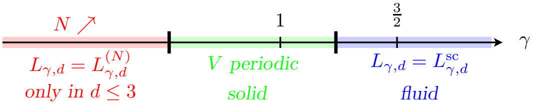
- $\simeq N$ quantum particles bound together by their own nonlinear potential V_N
- $=$ nonlinear Hartree model with Tsallis-type entropy $\mathrm{tr}(\Gamma^{\frac{\gamma}{\gamma-1}})$, where $\Gamma =$ 1-PDM
- existence for a sequence $N_j \to \infty$ suggests a statistical mechanics behavior, with the particles forming a **large cluster growing with N** (scenario 3)

Conjecture (Frank-Gontier-ML '21)

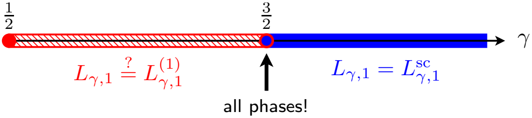
Normalize V_N in the manner $\|V_N\|_{L^\infty(\mathbb{R}^d)} = 1$. Then for $\gamma > \max\{0, 2-d/2\}$, V_N converges to a **periodic function** (possibly constant) which is an "optimizer" of $L_{\gamma,d}$.



Speculative phase diagram:



$\gamma = \frac{3}{2}$ in $d = 1$: an integrable system related to KdV equation

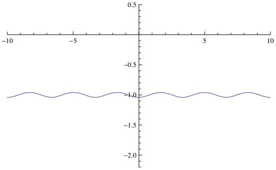


Theorem (Lieb-Thirring '76)

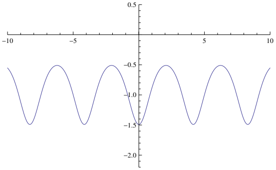
We have $L_{\frac{3}{2},1} = L^{(N)}_{\frac{3}{2},1} = L^{sc}_{\frac{3}{2},1} = \dfrac{3}{16}$*, with optimizers* V_N *for all* $N \geq 1$ *(KdV solitons).*

Theorem (Periodic optimizers, Frank-Gontier-ML '21)

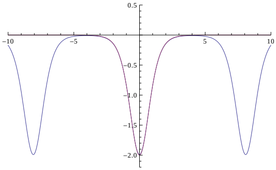
For all $0 < k < 1$*, the* $\ell = 2K(k)$ *periodic Lamé potential* $V(x) = 2k^2 \mathrm{sn}(x|k)^2 - 1 - k^2$ *is also an optimizer of* $L_{\frac{3}{2},1}$*. Here sn and* K *are the Jacobi elliptic function and complete elliptic integral of the first kind, with modulus* k*.*



$k = 0.2$ ($\ell = 3.32$)



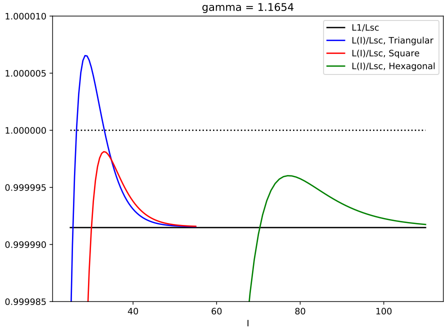
$k = 0.7$ ($\ell = 4.15$)

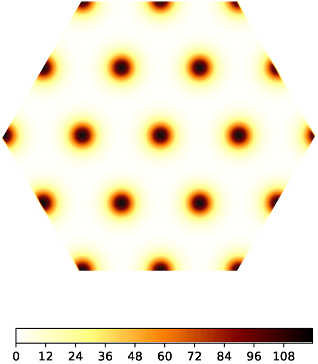


$k = 0.995$ ($\ell = 8.08$)

Numerics in 2D

In $d = 2$ we found **periodic potentials which beat both** $L^{(1)}_{\gamma,2}$ and $L^{\rm sc}_{\gamma,2}$





Important technical difficulties:

- very small difference between the lattices and the fluid
- binding energy really seems exponentially small, hard to catch
- need very high precision and the problem is nonlinear

Conclusion

$$\sum_{n=1}^{N} |\lambda_n(-\Delta+V)|^\gamma \leq L^{(N)}_{\gamma,d} \int_{\mathbb{R}^d} V(x)_-^{\gamma+\frac{d}{2}}\,dx, \qquad \gamma \begin{cases} \geq \frac{1}{2} & \text{for } d=1 \\ > 0 & \text{for } d=2 \\ \geq 0 & \text{for } d\geq 3 \end{cases}$$

- **Lieb-Thirring inequality**
  - estimate on $N \leq \infty$ lowest eigenvalues of $-\Delta + V(x)$
  - is the dual of Gagliardo-Nirenberg for $N=1$
- **Best constant $L_{\gamma,d}$**
  - is sometimes equal to Gagliardo-Nirenberg $(N=1)$
  - is often not given by any finite N
  - statistical mechanics problem for infinitely many fermions with nonlinear attraction
- **Optimal potential**
  - probably extended over the whole space $\mathbb{R}^d$
  - could be periodic
  - is constant for $\gamma \geq 3/2$
  - all possibilities happen at the special point $\gamma = \frac{3}{2}$ in $d=1$